2016

2.a.(i) Define a metric space. If A is a non-empty set and if f is a real valued function defined on $A\times A$ such that (i) $f(x,y)=0$ iff $x=y$ and (ii) $f(x,y) \le f(x,z)+f(y,z)$, $\forall\, x,y,z \in A$. Then prove that f is a metric ~~space~~ on A. (3)

Ans: Suppose X be a non-empty set then a function called distance function $d: X\times X \to \mathbb{R}$ satisfing following axioms is called a metric on X.

(i) $d(x,y) \geq 0$ for $(x,y) \in X\times X$ or $\forall\, x,y \in X$.

(ii) $d(x,y) = d(y,x)$, $\forall\, x,y \in X$, symmetric Property.

(iii) $d(x,z) \le d(x,y) + d(y,z)$, $\forall\, x,y,z \in X$, Triangular inequality.

And in that case the pair (X,d) is called a metric space.

We have, $f(x,y) = 0$ iff $x=y$ $\longrightarrow$ (i)

$f(x,y) \le f(x,z) + f(y,z)$, $\forall\, x,y,z \in A$ $\longrightarrow$ (ii)

From (ii) applying for x,x,y, we get

$$f(x,x) \le f(x,y) + f(x,y)$$

or, $f(x,x) \le 2f(x,y)$

or, $2f(x,y) \geq 0$ [Using (i)]

or, $f(x,y) \geq 0$.

We now show that $f(x,y) = f(y,x)$.

Again from (ii) applying for y,x,y, we have

$$f(y,x) \le f(y,y) + f(x,y) \longrightarrow \text{(iii)}$$

or, $f(y,x) \le f(x,y)$.

and from (ii) applying for x,y,x, we have

$$f(x,y) \le f(y,x). \longrightarrow \text{(iv)}$$

From (iii) and (iv), we get $f(x,y) = f(y,x)$.

$\therefore (A,f)$ is a metric space. i.e. f is a metric on A.

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2.a.(ii) Let, (X,ρ) and (Y,σ) be metric spaces and let $f: X \to Y$ be continuous at x_0; $x_0 \in X$. Prove that for every sequence $\{x_n\}$ in X which converges to x_0, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. (3)

Ans: Since, f is continuous at $x_0 \in X$ and let $\lim_{n\to\infty} x_n = x_0 \in X$.

i.e. $\rho(x_n, x_0) \to 0$ as $n \to \infty$.

Given $\epsilon > 0$, by continuity of f at x_0, we find a +ve δ such that $\sigma(f(x), f(x_0)) < \epsilon$, whenever $\rho(x, x_0) < \delta$.

Hence, we find index N such that $\rho(x_n, x_0) < \delta$, $\forall n \geq N$

So, $\sigma(f(x_n), f(x_0)) < \epsilon$, $\forall n \geq N$

i.e. $\lim_{n\to\infty} f(x_n) = f(x_0)$.

2.(b)(i) Define a closed set in a metric space (X,d). If $\{F_i : i \in I\}$ is any collection of closed set in X, then prove that intersection $\bigcap_{i\in I} F_i$ is a closed set.
Show by an example that union of finite no. of closed set may not be closed. (3)

Ans: Defn: A sub-set G of a metric space (X,d) is called a closed set if every limiting point of G belongs to G.

e.g. The real number space $\mathbb{R}$ with usual metric, every closed interval is a closed set.

■ Let, $F = \bigcap_{i\in I} F_i$, Then $X \setminus F = X \setminus \bigcap_{i\in I} F_i$

$= \bigcup_{i\in I} (X \setminus F_i)$ is an open set

$\because$ Each $X \setminus F_i$ is open.

Hence, $X \setminus F$ is open and hence, F is closed.

i.e. $\bigcap_{i\in I} F_i$ is a closed set.

■ ~~In a similar way,~~

Ex: In $(\mathbb{R}, d)$, where d is usual metric.

consider, for every natural number n, the closed interval $[1/n, 1]$ is a closed set in the space $\mathbb{R}$ with usual metric.

$\bigcup_{n=1}^{\infty} [1/n, 1] = (0, 1]$, it is not a closed set.

(ii) Let, (X, d) be a metric space. Show that every Cauchy sequence in X is bounded. Show by an example that a bounded sequence may not be a cauchy sequence. (3)

Ans: By def^n, given any $\varepsilon > 0$, $\exists$ an integer N such that

$$|a_n - a_m| < \varepsilon, \forall n, m \geq N$$

So, $$|a_n| - |a_m| \leq |a_n - a_m| < \varepsilon, \forall n, m \geq N$$

Taking $m = N$ and transposing, we have

$$|a_n| < |a_N| + \varepsilon, \forall n \geq N$$

Thus, $\forall n$ $$|a_n| \leq \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, |a_N| + \varepsilon\}.$$

A abound for $\{a_n\}$ is, then $M = \max\{|a_1|, \dots, |a_{N-1}|, |a_N| + \varepsilon\}$

$$|a_n| \leq M. \quad , \forall n \geq N.$$

Hence, every cauchy's sequence in X is bounded.

■ Consider a sub-set A of l_2 consisting an element $x_1 = \{1, 0, 0, \dots\}$, $x_2 = \{0, 1, 0, \dots\}$, $x_3 = \{0, 0, 1, \dots\}$, ...

Then $\rho(x_i, x_j) = \sqrt{2}$, for $i \neq j$

This shows that A is bounded but no sub-sequence ~~as~~ of $\{x_i\}$ can converge. So, $\{x_i\}$ is not a Cauchy ~~s~~ sequence.

Q. (i) If x, y, z, w are points in a metric space (X, d). Prove that $|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$. (5)

Ans: Since, (X,d) be a metric space. Then using triangle inequality for the points x, y, z, w, we get

~~$d(x,y) \leq d(x,z) + d(y,z)$ $d(z,w) + d(w,y)$ → (i)~~

~~Again using the points y, z, w we get~~

~~$d(y,w) \leq d(y,z) + d(z,w)$ → (ii)~~

~~Now, $|d(x,y) - d(z,w)| = |d(x,y) - d(y,z) + d(y,z) - d(z,w)|$.~~

~~$\leq |d(x,y) + d(y,z)| + |d(y,z) + d(z,w)|$.~~

~~$d(x,y) \leq d(x,z) + d(z,w) + d(w,y)$.~~

~~or, $d(x,y)$ — $d(z,w) \leq$~~

$d(x,y) \leq d(x,z) + d(z,w) + d(w,y)$ → (1)

or, $d(x,y) - d(z,w) \leq d(x,z) + d(w,y)$.

Similarly, $d(z,w) \leq d(z,x) + d(x,y) + d(y,w)$.

or, $d(z,w) - d(x,y) \leq d(x,z) + d(w,y)$ $[\because d(x,z) = d(z,x)$ & $d(y,w) = d(w,y)]$

or, $-(d(x,y) - d(z,w)) \leq d(x,z) + d(w,y)$

or, $d(x,y) - d(z,w) \geq -(d(x,z) + d(w,y))$ → (2)

From (1) and (ii) we get

$|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$. (Proved)

(ii) Let, (X,d) be a metric space and let A be a non-empty sub-set of X. Let, $f : X \to \mathbb{R}$ be defined by $f(x) = d(x,A), \forall x \in X$, where $d(x,A)$ denotes the distance of x from A. Show that f is continuous.

Proof: The function f is defined by, $f(x) = d(x,A) = \inf\{d(x,a) : a \in A\}, \forall x \in X$.

Now $\forall x, y \in A$ and $a \in A$, we get by triangle inequality,

$d(x,a) \leq d(x,y) + d(y,a)$.

i.e. $d(y,a) \geq d(x,a) - d(x,y)$.

$\geq \inf\{d(x,a) : a \in A\} - d(x,y)$.

$= d(x,A) - d(x,y)$.

Since, this is true for all $a \in A$, we get

$\inf\{d(y,a) : a \in A\} \geq d(x,A) - d(x,y)$.

i.e. $d(y,A) \geq d(x,A) - d(x,y)$.

or, $d(x,A) - d(y,A) \leq d(x,y)$.

Since, x and y are two arbitrary number choosen from X, then interchanging x and y, we get from the above inequality,

$d(y,A) - d(x,A) \leq d(y,x)$.

Combining the above two inequality, we find

$$|d(y,A) - d(x,A)| \leq d(x,y).$$

Let, $\varepsilon > 0$ be arbitrary small, we can find a $\delta > 0$ such that

$|f(x) - f(y)| < \varepsilon$ whenever $d(x,y) < \delta$.

i.e. $|d(x,A) - d(y,A)| < \varepsilon$ whenever $d(x,y) < \delta$

by setting $\delta = \varepsilon$.

Thus the function f is continuous.

5. a.(i) Let, (X,d) and (Y,ρ) be two metric spaces and let $f: X \to Y$. If X is compact and f is continuous then show that $f(x)$ is compact, where $f(x)$ is the range of f. (6)

Proof: Let, $\{B_\alpha\}$ be an open cover of $f(x)$ in Y, then let us put

$B_\alpha = f(x) \cap H_\alpha$, where H_α is an open set in (Y,ρ)

So, consider new open sets $\{H_\alpha\}_{\alpha\in\Delta}$ in (Y,ρ).

Since, f is continuous, so $f^{-1}(H_\alpha)$, for each $\alpha \in \Delta$ is an open set in (X,d) such that $f(x) \subseteq \bigcup_{\alpha\in\Delta} B_\alpha = \bigcup_{\alpha\in\Delta} (f(x) \cap H_\alpha)$

Since, $\{f^{-1}(H_\alpha)\}_{\alpha\in\Delta}$ is an open sets in (X,d),

Thus, $\{f^{-1}(H_\alpha)\}_{\alpha\in\Delta}$ becomes an open cover in X.

So by compactness of X, we have a finite sub-cover, say $\{f^{-1}(H_1)\}, \{f^{-1}(H_2)\}, \dots, \{f^{-1}(H_n)\}$, so that we have

$$X \subseteq f^{-1}(H_1) \cup f^{-1}(H_2) \cup \dots \cup f^{-1}(H_n).$$

or $f(x) \subseteq \{H_1 \cup H_2 \cup \dots \cup H_n\}$

or, $f(X) \subseteq \{H_1 \cap f(x)\} \cup \{H_2 \cap f(x)\} \cup \dots \cup \{H_n \cap f(x)\}$.

$$= B_1 \cup B_2 \cup \dots \cup B_n.$$

Hence, $\{B_1, B_2, \dots, B_n\}$ becomes a finite sub-cover for $f(x)$ and hence, $f(x)$ is compact.

c.(i) Let (X, ρ) and (Y, σ) be two metric spaces and let $f: X \to Y$ be continuous. If X is connected then show that the range $f(X)$ is also connected. Hence show that if f is a real valued continuous function defined on closed interval $[a, b]$ such that $f(a) < f(b)$ then f ~~is~~ continuous assumes every value between $f(a)$ and $f(b)$. $(6+2=8)$

Proof: Let, $f: (X, \rho) \to (Y, \sigma)$ be a continuous.

we are to prove that for any connected set $A \subset X$, $f(A)$ is connected in (Y, σ).

case. I. If $A = \phi$, $f(A) = f(\phi) = \phi$. Since, all null set is considered as a connected, there is nothing to prove.

case. II. If A is a singleton set, $A = \{a\}$ (say). then $f(A) = \{f(a)\}$. is also a singleton set. In this case $f(A)$ is connected.

case. III. Let, $A \subset X$ contains atleast two points. We are to prove that $f(A)$ is connected in (Y, σ).

If possible let $f(A)$ is disconnected, then $\exists$ two non-empty set G_1 and G_2 open in (Y, σ) such that

$f(A) \subset G_1 \cup G_2$, $f(A) \cap G_1 \neq \phi$, $f(A) \cap G_2 \neq \phi$

but $f(A) \cap (G_1 \cap G_2) = \phi$

Since, G_1 and G_2 are open in the metric space (Y, σ) and f is continuous mapping, then $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are open in (X, ρ).

Also,
$$\begin{aligned} & A \cap (f^{-1}(G_1) \cap f^{-1}(G_2)) \\ = & f^{-1}(f(A)) \cap \{f^{-1}(G_1) \cap f^{-1}(G_2)\} \\ = & f^{-1}\{f(A) \cap (G_1 \cap G_2)\} \\ = & f^{-1}(\phi) \\ = & \phi. \end{aligned}$$

Since, $f(A) \subset G_1 \cup G_2$. it follows that

$$A \subset f^{-1}(G_1 \cup G_2) = f^{-1}(G_1) \cup f^{-1}(G_2)$$

Finally $f(A) \cap G_1 \neq \phi \Rightarrow f^{-1}(f(A) \cap G_1) \neq \phi$
$\Rightarrow A \cap f^{-1}(G_1) \neq \phi$

Similarly, $f(A) \cap G_2 \neq \phi \Rightarrow f^{-1}(f(A) \cap G_2) \neq \phi$
$\Rightarrow A \cap f^{-1}(G_2) \neq \phi$

Thus we can express $A = A_1 \cup A_2$, where $A_1 = A \cap f^{-1}(G_1) \neq \phi$ and $A_2 = A \cap f^{-1}(G_2) \neq \phi$

But, $A_1 \cap A_2 = \phi$

Consequently A is disconnected in (X, d), which is a contradiction. Since, A is connected.

Hence, $f(A)$ must be connected in (Y, σ). This complete the proof.

Since, $f(a) < f(b)$. , Let y be any real number satisfing $f(a) \leq y \leq f(b)$. The domain of the function f, being a closed interval $[a, b]$, is connected. So, its image $f([a,b])$ is also connected. and hence is an interval. ~~Now~~, $y \in$

Now, $y \in [f(a), f(b)] \Rightarrow y \in f([a,b])$. $[\because [f(a), f(b)] \subset f([a,b])]$

Hence, $\exists$ atleast one $x \in [a,b]$ such that $f(x) = y$.

5. b. (i) Let, (X, d) be a metric space. and let E be a subset of $(E \subset X)$. If x is compact and E is closed then prove that E is compact.

Proof: Let, E be a closed sub-set of a compact metric space (X, d)

Let, $\{G_\alpha\}_{\alpha \in \Delta}$ be any open cover for E. Then the family $\{G_\alpha\}_{\alpha \in \Delta} \cup (X \setminus E)$ is an extended family of open sets, now serves as a open cover for X.

By compactness of (X, d), we have a finite sub-cover of this family say $\{G_1, G_2, \ldots, G_n\}, (X \setminus E)\}$.

Therefore, $G_1, G_2, \ldots, G_n$ and $(X \setminus E)$ together shall be an open cover for E.

i.e. Given an open cover for E, $\exists$ a finite subcover for E.

Hence, E is compact.

5.d) (i) For any two real numbers x and y, define $\sigma(x,y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$. Show that σ is a metric on the set of real numbers.

Ans: For $x, y \in \mathbb{R}$, $\sigma(x,y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| \geq 0$

$= 0$ iff $x = y$.

Again for $x, y \in \mathbb{R}$, $\sigma(x,y) = \sigma(y,x)$ is trivial.

If for $x, y, z \in \mathbb{R}$,

$$\sigma(x,y) + \sigma(y,z) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| + \left|\frac{y}{1+|y|} - \frac{z}{1+|z|}\right|$$

$$\geq \left|\frac{x}{1+|x|} - \frac{y}{1+|y|} + \frac{y}{1+|y|} - \frac{z}{1+|z|}\right|.$$

$$= \left|\frac{x}{1+|x|} - \frac{z}{1+|z|}\right|.$$

$$= \sigma(x,z)$$

i.e. $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$.

Triangle inequality holds.

Hence, σ is a metric on the set of all real numbers.

~~(ii) Prove that every metric space is 1^{st} countable and that a metric space is 2^{nd} countable iff it is separable.~~

5)b.(ii) Let, $C[0,1]$ be the set of all real valued continuous functions on $[0,1]$. Define $\tau(f,g) = \int_0^1 |f(x) - g(x)|\,dx$, for $f, g \in C[0,1]$. Show that τ is a metric on $C[0,1]$.

Let, $f_n(x) = \sqrt{n}(1-nx)$, if $0 \leq x \leq \frac{1}{n}$

$= 0$, if $\frac{1}{n} < x \leq 1$.

Show that $\{f_n\}$ is a sequence in $C[0,1]$ and it converges to a function in $C[0,1]$ w.r.to the metric ρ, where $\rho(f,g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$ for $f, g \in C[0,1]$.

2+4+2=8.

Ans: ■ Since, $\forall x \in [0,1]$, $|f(x) - g(x)| \geq 0$

Hence, $\int_0^1 |f(x) - g(x)| dx \geq 0$

i.e. $\tau(f,g) \geq 0$.

~~Further if $f \equiv g$~~ , then $\tau(f,g) = 0$.

Let, $f, g \in C[0,1]$

$\Leftrightarrow \int_0^1 |f(x) - g(x)| dx = 0$, $\forall x \in [0,1]$

$\Leftrightarrow |f(x) - g(x)| = 0$. [$\because f-g$ is continuous in $[0,1]$].

$\Leftrightarrow f(x) = g(x)$.

$\Leftrightarrow f = g$.

Also for all $f, g \in C[0,1]$.

$$\tau(f,g) = \int_0^1 |f(x) - g(x)| dx = \int_0^1 |g(x) - f(x)| dx = \tau(g,f).$$

Let us consider any $f, g, h \in C[0,1]$. Then all functions $f-h$, $g-h$ and $f-g$ ~~is~~ are continuous.

$\because |f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$, $\forall x \in [0,1]$

$$\Rightarrow \int_0^1 |f(x) - h(x)| dx \leq \int_0^1 |f(x) - g(x)| dx + \int_0^1 |g(x) - h(x)| dx.$$

i.e. $\tau(f,h) \leq \tau(f,g) + \tau(g,h)$.

Hence, $(C[0,1], \tau)$ is a metric space.

■■■ Let, $\{f_n\}$ be a Cauchy's sequence of $C[0,1]$ w.r.to the 'sup' metric ρ

i.e. $\rho(f_n, f_m) \to 0$ as $n, m \to \infty$

or, $\sup_{0 \leq x \leq 1} |f_n(x) - f_m(x)| \to 0$ as $n, m \to \infty$.

So for given $\varepsilon > 0$, $\exists$ a index N satisfing

$\sup_{0 \leq x \leq 1} |f_n(x) - f_m(x)| < \varepsilon$, $\forall n, m \geq N$

So, $|f_n(x) - f_m(x)| \leq \sup_{0 \leq x \leq 1} |f_n(x) - f_m(x)| < \varepsilon$, $\forall n, m \geq N$ and $\forall x \in [0,1]$.

i.e. $|f_n(x) - f_m(x)| < \varepsilon$, $\forall n, m \geq N$ and $\forall x \in [0,1]$.

Thus the real sequence $\{f_n(x)\}$ is cauchy sequence and by Cauchy general principle of convergent it convergent.

■■ Let, X denotes the set $C[0,1]$, we consider ~~C[0,1]~~ a metric τ on X given by $\tau(f,g) = \int_0^1 |f(x) - g(x)| dx$, $\forall f, g \in C[a,b]$

For each $n \in \mathbb{N}$, we consider the function $f_n : [0,1] \to \mathbb{R}$ given by,

$f_n(x) = \sqrt{n}(1-nx)$, if $0 \le x \le \frac{1}{n}$

$= 0$, if $\frac{1}{n} < x \le 1$.

Obviously f_n is continuous and hence $f_n \in X$, $\forall n \in \mathbb{N}$

we find for $m < n$ ($m, n \in \mathbb{N}$)

$d(f_n, f_m) = \int_0^1 |f_n(x) - f_m(x)| dx$

$= \int_0^{1/n} |f_n(x) - f_m(x)| dx + \int_{1/n}^{1} |f_n(x) - f_m(x)| dx$
$- \sqrt{m}(1-mx)$

$= \int_0^{1/n} \sqrt{n}(1-nx) dx + \int_{1/n}^{1} 0 \, dx$.

$= \left[\sqrt{n}x - n\sqrt{n}\frac{x^2}{2}\right]_0^{1/n} + \left[\sqrt{m}x - m\sqrt{m}\frac{x^2}{2}\right]_0^{1/m}$

$= \frac{\sqrt{n}}{n} - n\sqrt{n} \cdot \frac{1}{2n^2} + \frac{\sqrt{m}}{m} - m\sqrt{m} \cdot \frac{1}{2m^2}$

$\cancel{\frac{n}{n\sqrt{n}} - \frac{n}{2\sqrt{n}} = \frac{1}{\sqrt{n}}\left(1 - \frac{n}{2}\right)}$

$= \frac{1}{\sqrt{n}} - \frac{1}{2\sqrt{n}} + \frac{1}{\sqrt{m}} - \frac{1}{2\sqrt{m}}$

$= \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}} \to 0$ as $n, m \to \infty$ ($\because n > m$).

Hence, $\{f_n\}$ is a Cauchy sequence.

1.b. Examine wheather the sequence $\{\bar{x}_n\}$, where $\bar{x}_n = (\frac{1}{n}, \frac{1}{2n})$, is a Cauchy sequence in $\mathbb{R}^2$ with its usual metric.

Ans: Here, $\bar{x}_n = (\frac{1}{n}, \frac{1}{2n})$.

Now, $d(x_n, x_m) = |x_n - x_m| = \sqrt{(\frac{1}{n} - \frac{1}{m})^2 + (\frac{1}{2n} - \frac{1}{2m})^2}$

$= (\frac{1}{n} + \frac{1}{m}) \sqrt{1 + \frac{1}{4}}$

$= \frac{\sqrt{3}}{2}(\frac{1}{n} + \frac{1}{m}) \to 0$ as $n, m \to \infty$.

$\therefore$ $\{x_n\}$ is a cauchy sequence in $\mathbb{R}^2$.

d. Examine wheather the set $\{(x,y): 0 < x < 1; x \text{ is rational}; x = y\}$. is complete in $\mathbb{R}^2$ with its usual metric.

Ans: Let $X = \{(x,y): 0 < x < 1; x \text{ is rational}, x = y\}$.
Consider a sequence $\{x_n\}$, where $x_n = \frac{1}{2^n}$, $n = 1, 2, 3 \dots$

Here, $d(x_n, x_m) = \left|\frac{1}{2^n} - \frac{1}{2^m}\right| \le \frac{1}{2^n} + \frac{1}{2^m} \to 0$ as $n, m \to \infty$

Also, $\lim_{n \to \infty} \frac{1}{2^n} = 0$.

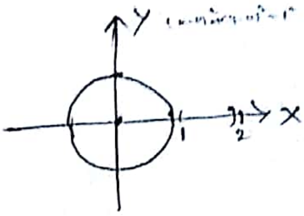
Hence, $\{x_n\}$ is a cauchy sequence on the space X.

Again consider a sequence $\{x_n\}$, where $x_n = (1 + \frac{1}{n})^n$, $n = 1, 2, 3, \dots$

Here, $\{x_n\}$ is cauchy but $\lim_{n \to \infty} x_n = \lim_{n \to \infty} (1 + \frac{1}{n})^n = e \notin X$

$\therefore X = \{(x,y) : 0 < x < 1 \;;\; x \text{ is rational} \;;\; x = y\}$ is not a complete metric space.

e. Let $S = \{(x,y) : x^2+y^2 = 1\} \cup \{(x,0) : 1 < x < 2\}$. Examine wheather is connected in $\mathbb{R}^2$ with its usual metric.



Ans: Let, $A = \{(x,y) : x^2+y^2=1\}$.

and $B = \{(x,0) : 1 < x < 2\}$.

Since, $S = A \cup B$, where

$A \neq \emptyset$ and $B \neq \emptyset$ But $A \cap B = \emptyset$,

where A and B are each open sets in S.

Hence, S is not connected with usual metric in $\mathbb{R}^2$.

c. Examine wheather the sequence $\{\bar{x}_n\}$, where $\bar{x}_n = \left(\sin\left(2+\frac{1}{n}\right), \log\left(3+\frac{1}{2n}\right)\right)$ is convergent in $\mathbb{R}^2$ with its usual metric.

Ans: Here $\bar{x}_n = \left(\sin\left(2+\frac{1}{n}\right), \log\left(3+\frac{1}{2n}\right)\right)$.

Now, $d(x_n, x_m) = |x_n - x_m|$ [consider d as a usual metric in $\mathbb{R}^2$].

$$= \sqrt{\left\{\sin\left(2+\frac{1}{n}\right) - \sin\left(2+\frac{1}{m}\right)\right\}^2 + \left\{\log\left(3+\frac{1}{2n}\right) - \log\left(3+\frac{1}{2m}\right)\right\}^2}$$

$$= \sqrt{\left(2\cos\frac{2+1/n+2+1/m}{2}\sin\frac{2+1/n-2-1/m}{2}\right)^2 + \left(\log\left(\frac{3+1/2n}{3+1/2m}\right)\right)^2}$$

$$\cancel{= \sqrt{\left[2\cos\left\{2+\tfrac{1}{2}\left(\tfrac{1}{n}+\tfrac{1}{m}\right)\right\}\right]^2 + \left\{\log\left(\frac{6+1/n}{6+1/m}\right)\right\}^2}}$$

$$= \sqrt{\left[2\cos\left\{2+\frac{1}{2}\left(\frac{1}{n}+\frac{1}{m}\right)\right\}\sin\frac{1}{2}\left(\frac{1}{n}-\frac{1}{m}\right)\right]^2 + \left\{\log\left(\frac{6+1/n}{6+1/m}\right)\right\}^2}$$

$\rightarrow 0$ as $n, m \rightarrow \infty$.

$\therefore \{x_n\}$ is a cauchy sequence. A

~~Hence convergent.~~